

4 Equilibrium

 Name

Learning objectives

- List the main assumptions of game theory
- Define and formulate strategic games
- Define Nash equilibrium, explain its applications and find the solution
- Define basic microeconomics concepts such as producer and consumer surplus, market equilibrium and social welfare
- Explain the difference between centralized operation and competition in power systems
- Explain the concepts of perfect and imperfect competition. Formulate problems that model these two situations and solve them
- Define, formulate and solve models with some players behaving as price-takers and others as Nash-Cournot

Bibliography

- Basic reading:
 - Osborne, Martin J., and Ariel Rubinstein. A course in game theory. MIT press, 1994 (chapters 1 and 2).
 - Gabriel, S., Conejo, A., Fuller, D., Hobbs, B., and Ruiz, C. Complementarity modeling in energy markets. Springer, 2012 (chapter 3).
 - Kirschen, D. S., and G. Strbac. Fundamentals of power system economics, 2004 (chapters 1 and 2).
 - Stoft, Steven. Power System Economics: Designing Markets for Electricity, Wiley-IEEE Press, 2002 (chapter 1-1).
- Further reading:
 - Neumann, Ludwig Johann, and Oskar Morgenstern. Theory of games and economic behavior. Vol. 60. Princeton, NJ: Princeton university press, 1947.
 - Nash, John F. Equilibrium points in n-person games. Proceedings of the national academy of sciences 36, no. 1 (1950): 48-49.



Game theory concepts



Strategic games



Nash equilibrium



Supply curve and producer surplus



Demand curve and consumer surplus



Market equilibrium

Example 4-1: Market equilibrium

We consider a linear inverse demand function $p = \alpha - \beta q$ and a linear supply function $p = \gamma + \delta q$, with $\alpha, \beta, \gamma, \delta \geq 0$. Determine, for $\alpha = 50$, $\beta = 0.2$, $\gamma = 15$, $\delta = 0.5$, the equilibrium price and quantity as well as the producer and consumer surplus.



Social welfare

Example 4-2: Equilibrium as maximization of social welfare

We consider a linear inverse demand function $p = \alpha - \beta q$ and a linear supply function $p = \gamma + \delta q$, with $\alpha, \beta, \gamma, \delta \geq 0$. Formulate an optimization problem that maximizes the social welfare with q as the decision variable. Express the optimality condition of such optimization problem and solve it for $\alpha = 50$, $\beta = 0.2$, $\gamma = 15$, $\delta = 0.5$.

In Example 1-3 an economic dispatch problem was formulated to minimize the production cost of satisfying a constant demand d . In this example, you are asked to:

- a) Formulate a similar problem to maximize the social welfare considering N^G generating units with a production cost function $\mathcal{C}_g(x_g)$ and capacity $\bar{x}_g$. The linear inverse demand function is formulated as $f_d^{-1}(d)$, where d is now a variable representing the demand level. Use $\underline{\eta}_g, \bar{\eta}_g$ as the dual variables corresponding to the production limits of the unit g , and λ as the dual variable corresponding to the balance equation.
- b) Formulate the KKT of the problem above with $\mathcal{C}_g(x_g)$ and $f_d^{-1}(d)$ convex and concave functions, respectively.
- c) What is the optimality condition if the limits of the units are disregarded?



Centralized operation vs. competition in power systems

- a) Formulate an optimization problem to maximize the profit of a price-taker power producer g with a production cost function $\mathcal{C}_g(x_g)$ and capacity $\bar{x}_g$. Electricity price is denoted by p . (Similar to Example 1-2).

Example 4-5: Equilibrium of price-taker producers

Formulate the equilibrium model of N^G price-takers producers. Assume that the inverse demand function is given by $f_d^{-1}(d)$. Compare this formulation with the one in Example 4-3.

Imperfect competition: monopoly

Example 4-6: Monopoly model

Formulate a profit maximization model (similar to Example 4-4) assuming that all producers merge into one single firm that anticipates the effect of the total production on the market price, which is formulated as the inverse demand function $f_d^{-1}(d)$. Use $\mathcal{C}_g(x_g)$ as the production cost function of the units, $\underline{\eta}_g$, $\bar{\eta}_g$ as the dual variables corresponding to the production limits of the units, and λ as the dual variable corresponding to the balance equation.

Determine the KKT corresponding to the previous optimization problem.

If the limits of the units are disregarded, what can we say now about λ ?

 Solving Examples 4-3 and 4-6 with GAMS (Example4-3.gms & Example4-6.gms)

Use GAMS to solve the optimization problem corresponding to Examples 4-3 and 4-6 for quadratic production cost functions ($C_g(x_g) = a_g x_g^2 + b_g x_g + c_g$) and linear inverse demand function ($f_d^{-1}(d) = \alpha - \beta d$). Use the following data: $\alpha = 70$, $\beta = 0.1$,

Unit	$\underline{x}_g$	$\bar{x}_g$	a_g	b_g	c_g
g_1	0	250	0.01	5	100
g_2	0	150	0.02	10	100
g_3	0	100	0.03	16	100

and fill in the following table with the results:

	Perfect competition (Example4-3.gms)	Monopoly (Example4-6.gms)
d		
price		
λ		
social welfare		
producer surplus		
consumer surplus		

Comment the results above.

Imperfect competition: Nash-Cournot

Example 4-7: Nash-Cournot model

Formulate a profit maximization model of one power producer with a convex cost function $\mathcal{C}_g(x_g)$ that knows the inverse demand function $f_d^{-1}(d)$. You can refer to the rest of power producer with the index $\tilde{g}$ and denote the guesses about their production level as $x_{\tilde{g}}$. *Tip: try to avoid the use of the total demand d in the formulation.*

Determine the KKT corresponding to the previous optimization problem.

Solving Example 4-7 with GAMS (Example4-7.gms)

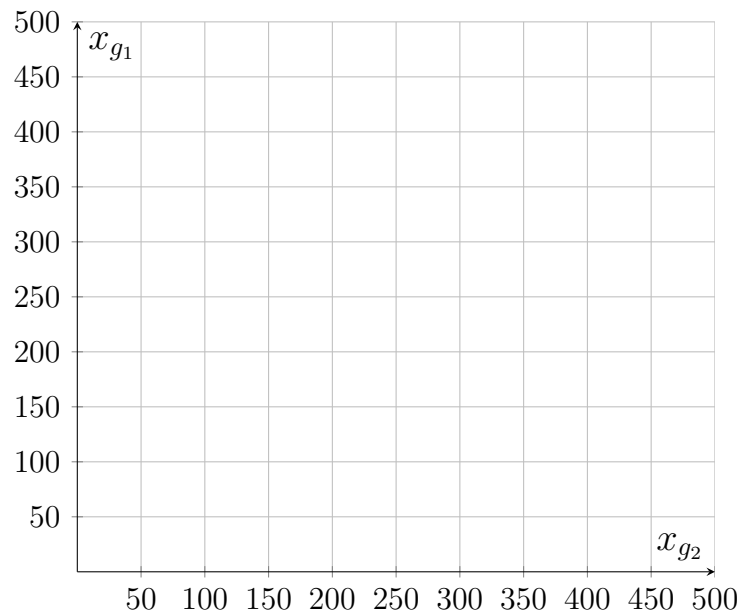
Use GAMS to solve the optimization problem corresponding to Example 4-7 for quadratic production cost functions ($\mathcal{C}_g(x_g) = a_g x_g^2 + b_g x_g + c_g$) and linear inverse demand function ($f_d^{-1}(d) = \alpha - \beta d$). Complete the table below using the following data $a_{g1} = 0.01$, $b_{g1} = 5$, $c_{g1} = 100$, $\bar{x}_{g1} = 500$, $\alpha = 70$, $\beta = 0.1$.

$\sum_{\tilde{g}} x_{\tilde{g}}$	0	100	200	300	400	500
x_{g1}						

Use the same code, fill in the table below for a different producer g_2 with the following data: $a_{g2} = 0.02$, $b_{g2} = 10$, $c_{g2} = 100$, $\bar{x}_{g2} = 500$, $\alpha = 70$, $\beta = 0.1$.

$\sum_{\tilde{g}} x_{\tilde{g}}$	0	100	200	300	400	500
x_{g2}						

If g_1 and g_2 are the only producers in the system, sketch below the optimal quantity produced by g_1 as a function of the production of g_2 , and the optimal quantity produced by g_2 as a function of the production of g_1 . Is there an equilibrium?



Example 4-8: Nash-Cournot equilibrium

Formulate a complementarity model to determine the equilibrium of several power producers using their corresponding KKT conditions. Use d when necessary and compare with the perfect-competition formulation obtained in Example 4-5.

Solving Example 4-8 with GAMS (Example4-8.gms)

Use GAMS to solve the complementarity problem of Example 4-8 for the data corresponding to g_1 and g_2 provided in Example 4-7. Is the equilibrium close to the one you inferred graphically?

Comparing Nash-Cournot model

Use the code Example4-8.gms to compare perfect competition, monopoly and Nash-Cournot models. Use the following data: $\alpha = 70$, $\beta = 0.1$,

Unit	$\underline{x}_g$	$\bar{x}_g$	a_g	b_g	c_g
g_1	0	250	0.01	5	100
g_2	0	150	0.02	10	100
g_3	0	100	0.03	16	100

and fill in the following table with the results:

	Perfect competition (Example4-3.gms)	Nash-Cournot (Example4-8.gms)	Monopoly (Example4-6.gms)
x_{g1}			
x_{g2}			
x_{g3}			
d			
price			
social welfare			
producer surplus			
consumer surplus			

Comment the results above.

Example 4-9: Nash-Cournot equilibrium as optimization problem

Being $\mathcal{C}_g(x_g) = a_g x_g^2 + b_g x_g + c_g$ and $f_d^{-1}(d) = \alpha - \beta d$, reformulate the complementarity model of Example 4-8. *Tip: replace d by $\sum_g x_g$.*

Try to reformulate the complementarity problem above as a single optimization problem.

Solving Example 4-9 (Example4-9.gms)

Use GAMS to solve the optimization problem formulated in Example4-9. Check that the solution is the same as that obtained by solving Example4-8.gms.

Example 4-10: Mixed-behaviors (complementarity model)

Formulate a complementarity problem similar to that in Example 4-8 considering that the producers $g \in G_{NC}$ behave as Nash-Cournot players, and the producers $g \in G_{PT}$ behave as price-takers. Use $\mathcal{C}_g(x_g)$ and $f_d^{-1}(d)$ as the production cost function and the inverse demand function, respectively.

Solving Example 4-10 (Example4-10.gms)

Use GAMS to solve the optimization problem formulated in Example 4-10 with the same data as in Example4-8.gms assuming that g_1 behaves as Nash-Cournot, and g_2 and g_3 as price takers. Fill in the following table with the results.

Mixed behaviours (g_1 as Cournot, g_2 and g_3 as price-takers)	
x_{g1}	
x_{g2}	
x_{g3}	
d	
price	
social welfare	
producer surplus	
consumer surplus	

Example 4-11: Mixed-behaviours (optimization model)

Formulate an optimization problem similar to that in Example 9 considering that the producers $g \in G_{NC}$ behave as Nash-Cournot players, and the producers $g \in G_{PT}$ behave as price-takers. Use $\mathcal{C}_g(x_g) = a_g x_g^2 + b_g x_g + c_g$ and $f_d^{-1}(d) = \alpha - \beta d$ as the production cost function and the inverse demand function, respectively.

Solving Example 4-11 (Example4-11.gms)

Use GAMS to solve the optimization problem formulated in Example 4-11 with the same data as in Example4-8.gms assuming that g_1 and g_2 behave as Nash-Cournot, and g_3 as price taker. Fill in the following table with the results.

Mixed behaviours (g_1 and g_2 as Cournot, g_3 as price-taker)	
x_{g1}	
x_{g2}	
x_{g3}	
d	
price	
social welfare	
producer surplus	
consumer surplus	

📌 Example 4-12: Comparing results of mixed-behaviours models

Fill in the following table with the results of Example4-3.gms, Example4-8.gms, Example4-10.gms, and Example4-11.gms where P-T and N-C stand for price-taker and Nash-Cournot players, respectively. Comment on the results.

g_1	P-T	N-C	P-T	P-T	N-C	N-C	P-T	N-C
g_2	P-T	P-T	N-C	P-T	N-C	P-T	N-C	N-C
g_3	P-T	P-T	P-T	N-C	P-T	N-C	N-C	N-C
x_{g1}								
x_{g2}								
x_{g3}								
d								
price								
SW								
PS								
CS								
profit g_1								
profit g_2								
profit g_3								

Example 4-13: Equilibrium of strategies

Using the results of Example 4-12 and assuming that g_3 behaves as price-taker, fill in the following table with the profits of g_1 , g_2 and g_3 . Is there an equilibrium of strategies between g_1 and g_2 ?

$g_1 \backslash g_2$	P-T	N-C
P-T	_____/_____/_____	_____/_____/_____
N-C	_____/_____/_____	_____/_____/_____

Using the results of Example 4-12 and assuming that g_3 behaves as Nash-Cournot, fill in the following table with the profits of g_1 , g_2 and g_3 . Is there an equilibrium of strategies between g_1 and g_2 ?

$g_1 \backslash g_2$	P-T	N-C
P-T	_____/_____/_____	_____/_____/_____
N-C	_____/_____/_____	_____/_____/_____

Looking at the results above, is there a Nash equilibrium in strategies between the three producers?

 Number of players in N-C equilibrium (Example4-14.gms and Example 4-15.gms)

Using the codes of Example4-3.gms and Example4-9.gms fill in the following table. The last column contains the results corresponding to a social welfare maximization problem with a single unit g_1 a production cost $C_{g1}(x_{g1}) = 5x_{g1}$, a capacity $\bar{x}_{g1} = 700$, and an inverse demand function $f_d^{-1}(d) = 70 - 0.1d$. The rest of the columns present the equilibrium results of N generating units with the same characteristics as g_1 that behave as Nash-Cournot.

	Nash-Cournot equilibrium						Competitive
	$N = 1$	$N = 2$	$N = 5$	$N = 10$	$N = 50$	$N = 100$	$N = 500$
d							
price							
SW							
PS							
CS							

Comment the results above.